8.1 The Dupuit Approximation

The Dupuit approximation is among the most powerful tools for treating unconfined flows. In fact, it is the only simple tool available to most engineers and hydrologists for solving such problems.

8.1.1 The Dupuit Assumptions

Dupuit (1863) developed a theory based on a number of simplifying assumptions resulting from the observation that in most ground water flows the slope of the phreatic surface is very small. In steady two-dimensional unconfined flow without accretion in the vertical $xz$ plane, the phreatic surface is a streamline. At every point $P$ along it, the specific discharge $q_s$ (fig. 8.1.1a) is given by Darcy's law:

$$q_s = -K \frac{d\varphi}{ds} = -K \frac{dz}{ds} = -K \sin \theta. \quad (8.1.1)$$

As $\theta$ is very small, Dupuit suggested that $\sin \theta$ be replaced by the slope $\tan \theta = \frac{dh}{dx}$. The assumption of a small $\theta$ is equivalent to assuming that equipotential surfaces are vertical (i.e., $\varphi = \varphi(x)$ is independent of $z$) and the flow essentially horizontal, or to assuming that we have a hydrostatic pressure distribution. Thus the Dupuit assumptions lead to the specific discharge expressed by:
\[ q_z = -K \frac{dh}{dx}; \quad h = h(x) \quad (8.1.2) \]

and to the total discharge through any vertical surface of width \( b \) (fig. 8.1.1b):

\[ Q_z = -Kbh(x) \frac{dh}{dx}. \quad (8.1.3) \]

It should be emphasized that all these assumptions may be considered as good approximations in regions where \( h \) is indeed small and the flow essentially horizontal.

The important advantage gained by employing the Dupuit assumptions is that the number of independent variables of the original problem \((x, z)\) has been reduced by one; in (8.1.3) \( z \) does not appear as an independent variable. We have here an extension of what is known as the hydraulic approach to fluid flows. In hydraulic flows, also called one-dimensional flows, we neglect in a nonuniform flow variations or changes in velocity, pressure, etc., transverse to the main flow direction. In figure 8.1.1 this is the \( x \) direction. At every cross-section perpendicular to the flow direction, conditions are expressed in terms of average values of velocity, density and other properties over the cross-section. The whole flow is considered as a single streamtube. This is the engineering approach that is most useful in treating pipe flow, flow in open channels, etc. Here we consider a single streamtube bounded by two streamlines: the phreatic surface and the impervious bottom; the cross-sections of interest are vertical. Thus, the usual dependent variable, the piezometric head \( \varphi = \varphi(x, z) \), is replaced by another variable \( h(x) \). Also, since at a point on the free surface \( \varphi = 0 \) and \( \varphi = h \), we assume that the vertical cross-sections are equipotential surfaces on which \( \varphi = h = \text{constant} \).

The Dupuit assumptions actually amount to neglecting the vertical flow component \( q_z = -K \frac{\partial \varphi}{\partial z} \). The value of \( q_z \) varies from \( q_z = 0 \) along the horizontal impervious boundary, to \( q_z = -K \frac{\partial \varphi}{\partial z} = -K \sin^2 \theta \) along the phreatic surface.